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Summability and Vector Amarts

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We show that an operator is absolutely summing if and only if it maps amarts into uniform amarts, from which we can deduce a theorem of A. Bellow and another of Edgar-Sucheston. We also show that the absolute value of a Banach lattice valued potential is a potential if and only if the lattice is an $A - M$ space from which we deduce that the L^1-bounded amarts form a Ricsz space if and only if the space is finite dimensional.

INTRODUCTION

Let E and F be Banach spaces. Denote by $\mathscr{L}(E, F)$ the space of continuous linear operators from E into F. We say that $T \in \mathscr{L}(E, F)$ is *absolutely summing* if T maps summable sequences in E into absolutely summable sequences in F.

Let $(\Omega, \mathscr{F}, P)$ be a probability space; denote by $L^1(E)$ the Banach space of Bochner integrable functions $X: \Omega \to E$ equipped with the classical norm $\int \| X(\omega)\|_E \, dP$.

It is known [5], that if $T \in \mathscr{L}(E, F)$, then T "lifts" to an operator $\tilde{T} \in \mathscr{L}(L^1(E), L^1(F))$ via:

$$(\tilde{T}X)(\omega) = T(X(\omega)) \quad \text{for} \quad \omega \in \Omega \quad \text{and} \quad X \in L^1(E).$$

If $\mu: \Omega \to E$ is an E-valued additive set function defined on an algebra $\mathcal{Q}$ of subsets of Ω, then $\tilde{T}\mu$ defined by $(\tilde{T}\mu)(A) = T(\mu(A))$ is an F-valued additive set function on $\mathcal{Q}$. If we define the "semivariation" of μ by: $\| \mu \|_{\mathscr{P}} = \sup_{\|f\| \leqslant 1}$ {variation $f(\mu)$}, one can prove as in [4] that $\| \mu \|_{\mathscr{P}} \leqslant 4 \sup_{A \in \mathcal{Q}} \| \mu(A)\|_E$.

Let now, $(\mathfrak{F}_n)_{n \in \mathbb{N}}$ be an increasing sequence of sub-σ-algebras of $\mathfrak{F}$ and let T denote the directed upward set of bounded stopping times. A sequence $(X_n)_n : \Omega \to E$ is called adapted if X_n is $\mathfrak{F}_n$-strongly measurable for each $n \in \mathbb{N}$.

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Definition 1. [3] An adapted sequence $(X_n)_n$ of E-valued random variables is called an E-valued *amart* if each X_n is Bochner integrable and $\lim_{\tau\in T}\int X_\tau$ exists in the norm topology of E.

The following, very useful stability property of amarts was established in [3]: Let $(X_n)_n$ be an E-valued amart. For each $\tau \in T$, set $\mu_\tau(A) = \int_A X_\tau$ for $A \in \mathfrak{F}_\tau$. Thus: the set $(\mu_\tau(A))_{\tau\in T}$ converges to a limit $\mu(A)$ for each $A \in \bigcup_{n\in\mathbb{N}} \mathfrak{F}_n$, and for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sigma \in T,\ \sigma \geqslant n_0 \Rightarrow \sup_{A\in\mathfrak{F}_\sigma} \|\mu_\sigma(A) - \mu(A)\| \leqslant \epsilon.$$

This property motivated A. Bellow [1] to introduce the following notion:

Definition 2. [1] $(X_n)_n$ is called a *uniform amart*, if for each $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that:

$$\sigma \in T, \qquad \sigma \geqslant n_0 \Rightarrow \text{variation}(\mu_\sigma - \mu \mid \mathfrak{F}_\sigma) \leqslant \epsilon.$$

Recently, this question was explicitly raised by D. H. Garling: Which type of operators map amarts into uniform amarts. The purpose of this note is to characterize these operators.

Theorem 1. *For an operator $T \in \mathscr{L}(E, F)$, the following properties are equivalent*:

(1) *T is absolutely summing.*

(2) *T maps E-valued amarts into F-valued uniform amarts.*

Proof. We prove first that if μ is an E-valued additive set function, and T is an absolutely summing operator, then:

$$\text{Variation}(T\mu) \leqslant K\,\|\mu\|_{\mathscr{P}} \leqslant 4K \sup_A \|\mu(A)\|_E \tag{*}$$

where K is a constant depending only on T.

Since T is absolutely summing, we have for every finite family $(x_i)_{1\leqslant i\leqslant n}$ in E:

$$\sum_{i=1}^{n} \|Tx_i\|_F \leqslant K \sup_{\substack{f\in E'\\ \|f\|\leqslant 1}} \left\{\sum_{i=1}^{n} |f(x_i)|\right\}$$

where K is a constant depending on T. Hence, for every finite sequence $(A_i)_{1\leqslant i\leqslant n}$ of disjoint sets in $\mathfrak{A}$:

$$\sum_{i=1}^{n} \|T\mu(A_i)\|_F \leqslant K \sup_{\substack{f\in E'\\ \|f\|\leqslant 1}} \left\{\sum_{i=1}^{n} |f\cdot\mu(A_i)|\right\}$$

and the inequality (*) follows.

The implication (1) $\Rightarrow$ (2) follows by applying (*) to $(\mu_\sigma - \mu \mid \mathfrak{F}_\sigma)$. (2) $\Rightarrow$ (1) If there exists a summable sequence $(x_n)_n$ in E such that $\sum_n \| Tx_n \| = \infty$, then we can find an increasing sequence of integers $(m_k)_k$ such that: $\sum_{n=m_k+1}^{m_{k+1}} \| Tx_n \| \geqslant 1$. Without loss of generality, and by multiplying some x_n's by coefficients smaller than 1, we can assume: $\sum_{n=m_k+1}^{m_{k+1}} \| Tx_n \| = 1$.

For every $k \in \mathbb{N}$, we can divide the interval $[0, 1]$, into $(m_{k+1} - m_k)$ subintervals $A_{k,n}$ such that the length of each $A_{k,n}$ is $\| Tx_n \|$. Let $\Omega = [0, 1]$ and P be the Lebesgue measure. We define a sequence of E-valued random variables by: $X_k : \Omega \to E$

$$X_k(\omega) = \sum_{n=m_k+1}^{m_{k+1}} \frac{x_n}{\| Tx_n \|} 1_{A_{k,n}}(\omega).$$

For every $k \in \mathbb{N}$, $\mathfrak{F}_k$ will be the σ-algebra $\sigma(X_1, X_2, \dots, X_k)$.

To show that $(X_k)_k$ is an amart, let σ be a bounded stopping time $\geqslant N$. For each $k \geqslant N$, let $B_{k,n} = A_{k,n} \cap \{\sigma = k\}$ we have

$$\int X_\sigma = \sum_{k \geqslant N} \int_{\{\sigma = k\}} X_k = \sum_{k \geqslant N} \sum_{n=m_k+1}^{m_{k+1}} P(B_{k,n}) \frac{x_n}{\| Tx_n \|}.$$

Since $B_{k,n} \subseteq A_{k,n}$, $P(B_{k,n})/\| Tx_n \| = \alpha_{k,n} \leqslant 1$. $\int X_\sigma = \sum_{k \leqslant N} \sum_{n=m_k+1}^{m_{k+1}} \alpha_{k,n} x_n$ which converges to zero since $(x_n)_n$ is summable.

$(TX_n)_n$ is an F-valued amart which is not a uniform amart, since if it was, one should have $\lim_{\sigma \in T} \int \| TX_\sigma \| = 0$ and by the real-valued amart convergence theorem, $(\| TX_k \|)_k$ must converge to zero a.e. But, it is easy to check that $\| TX_k(\omega) \|_F = 1$ for each $k \in \mathbb{N}$ and each $\omega \in \Omega$.

COROLLARY 1. *For a Banach space E, the following properties are equivalent:*

(1) *E is finite dimensional,*

(2) *Every E-valued amart is a uniform amart.*

Proof. Follows immediately from theorem (1) and the Dvoretzky-Rogers lemma.

COROLLARY 2. [2] *For a Banach space E, the following properties are equivalent:*

(1) *E is finite dimensional,*

(2) *Every E-valued amart of class(B) converges strongly a.e.*

Proof. The same as theorem (1), T being the identity operator.

Using the same ideas, we can show the following theorem proved in [5].

THEOREM 2. *For a Banach space E, the following properties are equivalent:*

(1) *E is finite dimensional,*

(2) *Every E-valued L^1-bounded amart converges weakly a.e.*

Proof. The x_n's being chosen as in theorem (1). For every $k \in \mathbb{N}$ we divide the interval $[0, 1/k]$ into $(m_{k+1} - m_k)$ disjoint subintervals $(A_{n,k})_{n=m_k+1}^{m_{k+1}}$ such that the length of each $A_{n,k}$ is $\| x_n \|/k$.

Let $A_k = \bigcup_{n=m_k+1}^{m_{k+1}} A_{n,k}$, $\Omega_k = [0, 1]$, λ_k the Lebesgue measure, Σ_k the σ-field of Borel set. $\Omega = \prod_k \Omega_k$, $\mathfrak{F} = \prod_k \Sigma_k$, $P = \prod_k \lambda_k$. For every $\omega \in \Omega$, ω_k will be the kth coordinate. We define:

$$X_k : \Omega \to E$$

$$X_k(\omega) = \begin{cases} \displaystyle\sum_{n=m_k+1}^{m_{k+1}} k \frac{x_n}{\| x_n \|} 1_{A_{n,k}}(\omega_k) \\ 0 \quad \text{if} \quad \omega_k \notin A_k . \end{cases}$$

Let σ be a bounded stopping time $\geqslant N$. For each $k \geqslant N$, let $B_{k,n} = A_{k,n} \cap \{\sigma = k\}$.

$$\int X_\sigma = \sum_{k \geqslant N} \int_{\{\sigma=k\}} X_k = \sum_{k \geqslant N} \sum_{n=m_k+1}^{m} P(B_{k,n})\, k \frac{x_n}{\| x_n \|}$$

which converges to zero.

Let now $B_k = \{\omega \in \Omega; \omega_k \in A_k\}$. The $(B_k)_k$ are independent and $\Sigma_k P(B_k) = \Sigma_k(1/k) = \infty$, hence we get from the theorem of Borel-Cantelli that almost all ω is in an infinite number of B_k's. But if $\omega \in B_k$, $\| X_k(\omega)\| \geqslant k$, thus for almost all ω, $\| X_k(\omega)\|$ is not bounded and $(X_k)_k$ cannot converge weakly.

Let now E be a Banach lattice. An important property of real valued amarts is that they are stable under the usual lattice operations. We now use the same summability arguments to show that it is not actually the case in infinite dimensional Banach lattices. We recall that (X_n) is said to be a *strong potential* if (X_n) converges to zero in the Pettis norm of $L^1[E]$.

THEOREM 3. *For a Banach lattice E the following properties are equivalent*:

(1) *E is isomorphic (as a topological vector lattice) to an A. M. space*

(2) *The absolute value of every E-valued strong potential is a strong potential.*

Proof. (1) $\Rightarrow$ (2) Let E be an $A - M$ space, E'' is then an $A - M$ space with unit; thus by Kakutani's representation theorem E'' is isomorphic to a $C(K)$ where K is a stonian compact. There is no loss then if we assume that $E = C(K)$. Let now, (X_n) be a strong potential, that is $\sup_{A \in F_\sigma} \| \int_A X_\sigma \| \to 0$. We claim that $\int | X_\sigma |$ norm converges to zero, and that $(| X_n |)$ is a strong potential.

Suppose not, then there exists $\epsilon > 0$, such that for every $n \in N$, there exists $\sigma_n \in T$, $\sigma_n \geqslant n$ and $t_n \in K$ and

$$\int | X_{\sigma_n}(w)(t_n)| \, dP(w) \geqslant \epsilon.$$

Let m_n be an integer such that $\sigma_n \leqslant m_n$. Clearly, $A = \{w; X_{\sigma_n}(w)(t_n) \geqslant 0\} \in F_{m_n}$ and

$$\int_A X_{\sigma_n}(w)(t_n) \, dP(w) \geqslant \epsilon/2.$$

Define the stopping time

$$\tau_{p,n} = \begin{cases} \sigma_n & \text{on } A \\ p \geqslant m_n & \text{on } A^c. \end{cases}$$

we have:

$$\left| \int_\Omega X_{\tau_{p,n}}(w)(t_n) \, dP(w) \right| = \left| \int_A X_{\sigma_n}(w)(t_n) \, dP(w) + \int_{A^c} X_p(w)(t_n) \, dP(w) \right|.$$

Since the third term goes to zero when $p \to \infty$, we may then find p_n large enough such that

$$\left| \int_\Omega X_{\tau_{n,p_n}}(w)(t_n) \right| \geqslant \epsilon/4$$

which is a contradiction.

(2) $\Rightarrow$ (1) Suppose now that E is not an $A - M$ space, hence by [7] there exists a summable sequence (x_n) in E such that $(| x_n |)$ is not summable. By the Orlisz-Pettis theorem, there exists f in E_+ such that $f(| x_n |)$ is not absolutely summable in $\mathbb{R}$. We may again construct an increasing sequence of integers (m_k) such that

$$\sum_{n=m_k+1}^{m_{k+1}} f(| x_n |) = 1 \qquad \text{for every} \quad k.$$

As above, divide for each k, the interval $[0, 1]$ into $(m_{k+1} - m_k)$ subintervals $(A_{k,n})$ such that the length of each $A_{k,n}$ is $f(| x_n |)$. Let $(\Omega, \mathscr{F}, P)$ be as in theorem (1) and define:

$$X_k(w) = \sum_{n=m_k+1}^{m_{k+1}} \frac{x_n}{f(| x_n |)} 1_{A_{k,n}}(w).$$

The same proof as above show that (X_k) is a strong potential; however $(| X_n |)$ is not since it is easy to check that $f(| X_n |(w)) = 1$ for each n and each $w \in \Omega$.

Corollary 1. *For a Banach lattice E, the following are equivalent:*

(1) *E is finite dimensional.*

(2) *The L^1-bounded amarts form a Ricsz space.*

Proof. We first show that if the absolute value of an L^1-bounded martingale is an amart then E is weakly sequentially complete. Consider an independent sequence (Y_n) of real random variables taking the values ± 1 with probabilities $1/2$ and $1/2$. Clearly, the sequence

$$X_n : (\Omega, \mathscr{F}, P) \to c_0$$

$$X_n = (Y_0, Y_1, \ldots, Y_n, 0, 0, \ldots)$$

is an L^1-bounded martingale. On the other hand, $|X_n| = (1, 1, \ldots, 1, 0, 0, \ldots)$ is not norm convergent to an element of c_0. Therefore c_0 is not a sublattice in E and E is weakly sequentially complete. It is well known that a weakly sequentially complete $A - M$ space is finite dimensional.

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